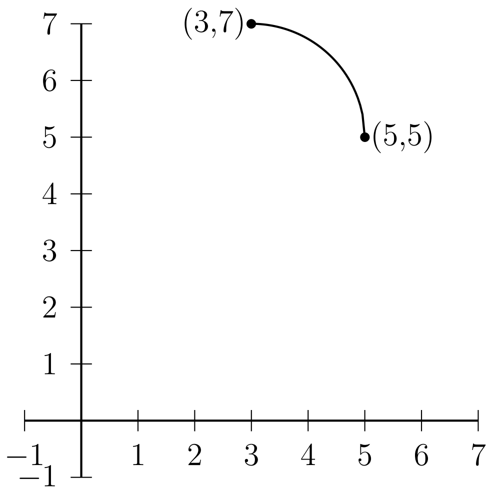
Exercise XI

1. Use the Intermediate Value Theorem and Rolle's Theorem to prove that
$$x^7 + 3x^5 + x^3 + x + 1 = 0$$
has exactly one real solution.

2. The graph of the function $y = f(x)$ is a quadrant of a circle as illustrated below. Determine $\int_5^7 f^{-1}(x)\mathrm{d}x$:



3. Let $f(x) = x$ and let P_n be the partition of the interval $[a, b]$ into n equal sub-intervals. Verify that
$$\lim_{n\to\infty} U(f, P_n) = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2$$
(Note: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.)

4. For each of the following functions $f(x)$, find $\int_{-2}^{2} f(x)\mathrm{d}x$:
(i)$f(x) = \begin{cases} 1, & x \le 1 \\ 2, & x > 1. \end{cases}$, (ii) $f(x) = |x|$

5. Let $F(x) = \int_2^x t^3 \mathrm{d}t$. Determine $F'(x)$.

6. Determine the Taylor polynomial of degree 8 around 0 for $\cos(x)$.

7. Determine the Taylor polynomial of degree 4 around 0 for $(1+x)^{\frac{1}{2}}$

8. Use the Taylor polynomial of degree 5 around 0 for $\sqrt{1+x}$ to estimate $\sqrt{2}$.

9. Show that the Taylor expansion around 0 for $\dfrac{1}{1-x}$ does not converge for $|x| > 1$.